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The classical limit for the Holstein–Primakoff representation in the soliton theory of Heisenberg chains

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Abstract. Soliton excitations in the anisotropic Heisenberg chain are studied in the classical limit by means of the Holstein–Primakoff boson representation, taking into account the complete series. The results obtained are equivalent to those obtained by the classical treatment of spins. The connection with different calculations using limited numbers of terms is established, as well as the relation to the generalised coherent-states method.

1. Introduction

Bosonic representations of the spin operators turn out to be a very suitable method for studying the solitary waves in magnetic systems, since they allow one to include quantum corrections in a systematic way. Prior to the study of quantum effects, one has to establish the strict correspondence between the classical solutions (solutions obtained by the treatment of spins as classical vectors) and the solutions obtained by means of boson representations. (For the spin coherent-state representation, such a correspondence was established by Balakrishnan and Bishop (1985).)

The boson representation particularly suitable for this approach is the Holstein–Primakoff (HP) (1940) representation. The pioneering work of Pushkarov and Pushkarov (1977) on this subject suffered from a highly inconsistent comparison of the terms (which in fact were of the same order of magnitude), so that the first consistent study of the solitons in the isotropic Heisenberg ferromagnet within the framework of the HP representation was performed by de Azevedo *et al* (1982). Following their work, we have extended the same approach to the anisotropic Heisenberg model (Škrinjar *et al* 1987). In fact, in both cases a limited number of terms in the Bose operators were kept (a 'truncated' representation); so one has to consider the terms whose contributions were not included.

The aim of this paper is to show that the inclusion of the whole series leads in the classical limit strictly to the classical results and enables one to establish clearly the scope of the quantum corrections, whose explicit calculation is postponed until a subsequent paper.

The structure of the paper is as follows: we introduce the Hamiltonian in § 2, and express it in terms of the HP representation using Glauber's coherent-state representation to evaluate the classical limit. The equations of motion are derived in § 3, while soliton characteristics are calculated for some particular cases in § 4, where a comparison

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with other results is performed. The problem of the connection between Glauber's representation and the spin coherent states initiated in the pioneering work of Radcliffe (1971) is discussed in the Appendix within the framework of the present results.

2. The Hamiltonian of the system

We shall study the anisotropic Heisenberg chain in an external field f, whose Hamiltonian can be put into the form

$$H - H_0 = -\mu f \sum_j (S_j^z - S) - J \sum_j (\mathbf{S}_j \cdot \mathbf{S}_{j+1} - S^2) - J\tau \sum_j (S_j^z S_{j+1}^z - S^2). \tag{1}$$

Here $\mathbf{S}_j$ denotes the spin of the jth ion, J is the exchange integral, τ is the anisotropy parameter, μ is the magnetic moment and a is the lattice constant.

Let us introduce the HP representation:

$$S_j^z = S - B_j^+ B_j \tag{2a}$$

$$S_j^- = (S_j^+)^+ \tag{2b}$$

$$S_j^+ = [2S(1 - B_j^+ B_j/2S)]^{1/2} B_j. \tag{2c}$$

B_j, B_j^+ are Bose operators satisfying

$$[B_i, B_j] = 0 \qquad [B_i, B_j^+] = \delta_{ij}.$$

We shall now take a closer look at the square root:

$$\left(1 - \frac{B_j^+ B_j}{2S}\right)^{1/2} = \sum_{K=0}^{\infty} \binom{\frac{1}{2}}{K} \left(-\frac{1}{2}\right)^K \left(\frac{B_j^+ B_j}{S}\right)^K.$$

The application of the coherent states demands the operator expression to be put into the form of a normal ordered product. Simple manipulation shows that one always has

$$(B_j^+ B_j)^K = B_j^{+K} B_j^K + a_{K-1} B_j^{+K-1} B_j^{K-1} + \ldots + a_2 B_j^{+2} B_j^2 + B_j^+ B_j \tag{3}$$

where a_n are coefficients that can be determined. This implies that any term of the type $B^{+K} B^K / S^K$ also has a correction arising from normal ordering of higher terms in the Bose operators, but with a coefficient at least of order $1/S^{K+1}$, or smaller. Such terms can be neglected in the classical limit ($S \to \infty$); so we have

$$\left(1 - \frac{B_j^+ B_j}{2S}\right)^{1/2} = \sum_{K=0}^{\infty} \binom{\frac{1}{2}}{K} \left(-\frac{1}{2}\right)^K \left(\frac{B_j^{+K} B_j^K}{S^K}\right)\left[1 + \mathrm{O}\left(\frac{1}{S}\right)\right] \approx \sum_{K=0}^{\infty} \binom{\frac{1}{2}}{K} \left(-\frac{1}{2}\right)^K \frac{B_j^{+K} B_j^K}{S^K}.$$

We still have to prove that the only remaining terms are those that we kept in the above expression.

This will be shown in the next step, which is the averaging of the Hamiltonian (1) over boson coherent states $|\alpha_j\rangle$ (Glauber 1963):

$$|\alpha\rangle = \prod_{j=1}^{N} |\alpha_j\rangle. \tag{4}$$

Looking at the expression for S^z

$$\langle\alpha|S_j^z|\alpha\rangle = S - \langle\alpha_j|B_j^+B_j|\alpha_j\rangle = S - |\alpha_j|^2 \tag{5}$$

where the α_j-values are coherent amplitudes ($\hat{B}|\alpha\rangle = \alpha|\alpha\rangle$) and comparing it with the classical expression

$$S^z = S\cos\theta$$

we note the correspondence

$$|\alpha_j|^2 = S(1-\cos\theta).$$

We see that $|\alpha_j|^2$ diverges for $S\to\infty$, but the quantity

$$\tilde{\alpha}_j = \alpha_j/\sqrt{S} \tag{6}$$

remains finite, and we shall use it in the future.

Next, we calculate

$$\begin{aligned}\langle\alpha|S_j^+|\alpha\rangle &= \sqrt{2S}\langle\alpha_j|\sqrt{1-\frac{B_j^+B_j}{2S}}B_j|\alpha_j\rangle = \sqrt{2S}\langle\alpha_j|\sum_{K=0}^{\infty}\binom{\frac{1}{2}}{K}\left(-\frac{1}{2}\right)^K \\ &\quad\times\left(\frac{B_j^+B_j}{S}\right)^K B_j|\alpha_j\rangle \simeq \sqrt{2S}\langle\alpha_j|\sum_{K=0}^{\infty}\binom{\frac{1}{2}}{K}\left(-\frac{1}{2}\right)^K\frac{B_j^{+K}B_j^K}{S^K}B_j|\alpha_j\rangle \\ &= \sqrt{2S}\sum_{K=0}^{\infty}\binom{\frac{1}{2}}{K}\left(-\frac{1}{2}\right)^K\frac{|\alpha_j|^{2K}}{S^K}\alpha_j \\ &= \sqrt{2S}\sqrt{1-\frac{|\alpha_j|^2}{2S}}\,\alpha_j \\ &= \sqrt{2}S\sqrt{1-\frac{|\tilde{\alpha}_j|^2}{2}}\,\tilde{\alpha}_j.\end{aligned} \tag{7}$$

This enables us to write down the complete Hamiltonian.

Since our aim is to study the classical limit $S\to\infty$, $\hbar\to 0$, we have to write the Hamiltonian in the correct dimensional form. Until now, we have used the system $\hbar = 1$ in order to simplify bosonic representation. For this reason, we shall treat spin operators as dimensionless and introduce the necessary factors of $\hbar$ (Jauslin and Schneider 1982, Jauslin 1982). This procedure gives

$$\begin{aligned}\langle\alpha|\hat{H}-H_0|\alpha\rangle &= \mu f\hbar S\sum_j|\tilde{\alpha}_j|^2 - J\hbar^2S^2\sum_j\left[\left(1-\frac{|\tilde{\alpha}_j|^2}{2}\right)\left(1-\frac{|\tilde{\alpha}_{j+1}|^2}{2}\right)\right]^{1/2} \\ &\quad\times(\tilde{\alpha}_{j+1}^*\tilde{\alpha}_j + \tilde{\alpha}_j^*\tilde{\alpha}_{j+1}) - (1+\tau)J\hbar^2S^2\sum_j(|\tilde{\alpha}_j|^2|\tilde{\alpha}_{j+1}|^2 - 2|\alpha_j|^2).\end{aligned} \tag{8}$$

We introduce the classical quantities

$$S_c = \lim_{\substack{\hbar\to 0\\ S\to\infty}}(\hbar S) \tag{9a}$$

$$\tilde{\mu} = \mu S_c \tag{9b}$$

$$\tilde{J} = JS_c^2 \tag{9c}$$

$$\langle\alpha|\hat{H} - H_0|\alpha\rangle = \tilde{\mu}f\sum_j |\tilde{\alpha}_j|^2 - \tilde{J}\sum_j [(1 - \tfrac{1}{4}|\tilde{\alpha}_j|^2)(1 - \tfrac{1}{2}|\tilde{\alpha}_{j+1}|^2)]^{1/2}$$
$$\times (\tilde{\alpha}^*_{j+1}\tilde{\alpha}_j + \tilde{\alpha}^*_j\tilde{\alpha}_{j+1}) - (1 + \tau)\tilde{J}\sum_j (|\tilde{\alpha}_j|^2|\tilde{\alpha}_{j+1}|^2 - 2|\tilde{\alpha}_j|^2). \quad (10)$$

The next step is to perform the continuum limit: $\tilde{\alpha}_j \to \tilde{\alpha}(x, t)$, $\tilde{\alpha}_{j\pm1} \to \tilde{\alpha}(x \pm a, t)$, $\Sigma_j \to (1/a)\int \mathrm{d}x$. We shall expand all quantities up to a^2. In this way, we obtain the following Hamiltonian density:

$$\langle\alpha|\hat{H} - H_0|\alpha\rangle = \frac{1}{a}\int \mathscr{H}\,\mathrm{d}x \quad (11a)$$

$$\mathscr{H} = h_2|\tilde{\alpha}|^2 - \tau\tilde{J}|\tilde{\alpha}|^4 + \tilde{J}a^2|\tilde{\alpha}_x|^2 + \tfrac{1}{4}\tilde{J}a^2(\tilde{\alpha}^2\tilde{\alpha}_x^{*2} + \tilde{\alpha}^{*2}\tilde{\alpha}_x^2) + \tfrac{1}{16}\tilde{J}a^2[|\tilde{\alpha}|^2/(1 - \tfrac{1}{2}|\tilde{\alpha}|^2)]$$
$$\times (2|\tilde{\alpha}_x|^2|\tilde{\alpha}|^2 + \tilde{\alpha}^2\tilde{\alpha}_x^{*2} + \tilde{\alpha}^{*2}\tilde{\alpha}_x^2) + \tfrac{1}{2}\tau\tilde{J}a^2[(\partial/\partial x)|\tilde{\alpha}|^2]^2 \quad (11b)$$

with $h_2 = \tilde{\mu}f + 2\tau\tilde{J}$. Here $\tilde{\alpha}$ stands for $\tilde{\alpha}(x, t)$ and the subscript denotes the corresponding partial derivative. This expression was obtained under the assumption of cyclic boundary conditions and the condition

$$|\tilde{\alpha}|^2_x(\pm\infty) = |\tilde{\alpha}|^2_{xx}(\pm\infty) = \ldots = 0.$$

3. Equations of motion

The usual treatment accepts $\mathscr{H}$ as the density of the classical Hamiltonian and treats $\tilde{\alpha}$ and $\tilde{\alpha}^*$ as the pair of conjugated canonical variables. In this case, one writes down the Hamiltonian equation of motion (with the classical transition included) as

$$\mathrm{i}S_c(\partial\tilde{\alpha}/\partial t) = \partial\mathscr{H}/\partial\tilde{\alpha}^* - (\partial/\partial x)(\partial\mathscr{H}/\partial\tilde{\alpha}^*_x). \quad (12)$$

Partial integrations performed in order to obtain $\mathscr{H}$ in the form (11b) help us now to avoid higher derivatives in (12). The resulting expression is

$$\mathrm{i}S_c\tilde{\alpha}_t = -h_2\tilde{\alpha} - 2\tau J|\tilde{\alpha}|^2\tilde{\alpha} - \tilde{J}a^2\tilde{\alpha}_{xx} - \tfrac{1}{4}\tilde{J}a^2(2\tilde{\alpha}|\tilde{\alpha}_x|^2 + \tfrac{1}{2}\tilde{\alpha}^2\tilde{\alpha}^*_{xx} - \tfrac{1}{2}\tilde{\alpha}^*\tilde{\alpha}_x^2)$$
$$- \tfrac{1}{16}\tilde{J}a^2\{\tilde{\alpha}(|\tilde{\alpha}|^2_x)/(1 - \tfrac{1}{2}|\tilde{\alpha}|^2)^2 + [2|\tilde{\alpha}|^2\tilde{\alpha}/(1 - \tfrac{1}{2}|\tilde{\alpha}|^2)]|\tilde{\alpha}|^2_{xx}\}$$
$$- \tau\tilde{J}a^2(2\tilde{\alpha}|\tilde{\alpha}_x|^2 + \tilde{\alpha}^2\tilde{\alpha}^*_{xx} + |\tilde{\alpha}|^2\tilde{\alpha}_{xx}). \quad (13)$$

Now we can introduce the usual substitution

$$\tilde{\alpha}(x, t) = A(x, t)\exp[\mathrm{i}\varphi(x, t) + \mathrm{i}\Omega t] \quad (14)$$

with A and φ real functions.

Separating the real and imaginary parts of (13), we can obtain several important results. First of all, let us look at the imaginary part of the equation:

$$-S_cA_t = \tilde{J}a^2(-2A_x\varphi_x - A\varphi_{xx} + 2A^2\varphi_x + \tfrac{1}{2}A^3\varphi_{xx}). \quad (15)$$

It is very important to note that there are no contributions from the anisotropic interaction (no terms proportional to τ) in (15). This means that all the consequences of this equation are valid in both the isotropic and the anisotropic case.

Further, we can see that the only terms which contribute to (15) are those of order $|\tilde{\alpha}|^4$, so that equations of this type obtained in calculations with up to four Bose operator terms are correct.

The real part of the equation gives

$$(-S_c\varphi_t - S_c\Omega)A = h_2A - 2\tau\tilde{J}A^3 - \tilde{J}a^2A_{xx} + \tilde{J}a^2A\varphi_x^2 + \tilde{J}a^2A^3\varphi_x^2$$
$$- \tfrac{1}{2}\tilde{J}a^2(1+4\tau)(AA_x^2 - A^2A_{xx}) - \tfrac{1}{8}\tilde{J}a^2[(4A^3A_x^2 + 2A^4A_{xx})/(1-\tfrac{1}{2}A^2)$$
$$+ A^5A_x^2/(1-\tfrac{1}{2}A^2)^2]. \tag{16}$$

We shall look for the solitary solution of the form

$$A(x,t) = A(x-vt) \qquad \varphi(x,t) = \varphi(x-vt) \qquad x - vt = \xi \tag{17}$$

where v is the soliton velocity.

The boundary conditions now become

$$A(\pm\infty) = A_\xi(\pm\infty) = A_{\xi\xi}(\pm\infty) = \ldots = 0.$$

Multiplying (15) by A, one can integrate it to obtain

$$\varphi_\xi = (V/2)[1/(1-\tfrac{1}{2}A^2)] \tag{18}$$

where

$$V = S_c v/\tilde{J}a^2. \tag{19}$$

Substituting (17) into (16), and multiplying the equation by A_ξ, we obtain

$$(\mathrm{d}/\mathrm{d}\xi)[\tfrac{1}{2}A_\xi^2 + \tfrac{1}{4}(1+4\tau)A_\xi^2A^2 + \tfrac{1}{8}A_\xi^2A^4/(1-\tfrac{1}{2}A^2) + (V^2/4)[(1-\tfrac{1}{2}A^2)]^{-1}$$
$$- \gamma_0A^2 - (\tau/a^2)A^2 + (\tau/2a^2)A^4] = 0 \tag{20}$$

with

$$\gamma_0 = (S_c\Omega + \tilde{\mu}f)/\tilde{J}a^2. \tag{21}$$

Integrating this, and applying boundary conditions to determine the integration constant, we arrive at

$$A_\xi^2[1 + 2\tau A^2(1-\tfrac{1}{2}A^2)] = \gamma_0A^2[1 + 2\tau/a^2\gamma_0 - V^2/4\gamma_0$$
$$- (A/2)(1 + 4\tau/\gamma_0a^2) + (\tau/2\gamma_0a^2)A^4]. \tag{22}$$

Following Tjon and Wright (1977), we introduce a new variable

$$\tfrac{1}{2}A^2 = (1-\cos\theta)/2 = \sin^2\beta \tag{23}$$

to obtain

$$\cos^2\beta\beta_\xi^2(1 + 4\tau\sin^2\beta\cos^2\beta) = \gamma_0\sin^2\beta[\cos^2\beta + (2\tau/\gamma_0a^2)\cos^4\beta - V^2/4\gamma_0]. \tag{24}$$

This equation was previously derived by Tjon and Wright (1977) in a completely classical treatment and by us (Kapor *et al* 1986) using a semi-classical approach.

4. Discussion of results

Let us first discuss our result for the isotropic chain ($\tau = 0$). We obtain

$$A_\xi^2 = \gamma_0A^2(1 - V^2/4\gamma_0 - A^2/2). \tag{25}$$

This expression, of course, agrees with the results of Tjon and Wright, but it is more interesting to note that it also agrees with the result of de Azevedo *et al* (1982). They obtained it by expanding the HP representation up to the terms with four Bose operators, under the assumption that coherent amplitudes satisfy the condition $\lim_{S\to\infty}[(1/2S)|\alpha|^2]\to 0$.

In fact, this quantity remains finite, since it becomes

$$(1/2S)|\alpha|^2 = \tfrac{1}{2}|\tilde{\alpha}|^2 = (1-\cos\theta)/2$$

in the classical limit; so it satisfies only a weaker condition $\frac{1}{2}|\tilde{\alpha}|^2 \leqslant 1$, which is sufficient to ensure the convergence of the series, neglecting terms of order $(1/S)|\tilde{\alpha}|^2$ but not the above-mentioned condition of de Azevedo *et al* (1982).

The reason that their result agrees with the correct expression is because, for $\tau = 0$, all terms of order higher than $\tilde{\alpha}^4(A^4)$ exactly cancel in (8). This is why the result obtained by this 'truncated' expansion turns out to be correct.

Single-soliton solutions and the system energy can be expressed in terms of the magnetisation and momentum of the system, following Tjon and Wright (1977):

$$M = \frac{1}{a}\int S_c(1-\cos\theta)\,dx = \frac{S_c}{a}\int A^2\,dx \tag{26}$$

$$P = \frac{1}{a}\int S_c(1-\cos\theta)\varphi_x\,dx = \frac{S_c}{a}\int A^2\varphi_x\,dx. \tag{27}$$

The single-soliton solution is

$$A^2 = 2\varphi_0^2/\cosh[(2/\Gamma)(x-vt)] \tag{28}$$

with the amplitude

$$\varphi_0^2 = \sin(Pa/4S_c) \tag{29}$$

and the soliton dimension

$$\Gamma = (Ma/2S_c)(1/\varphi_0^2). \tag{30}$$

The soliton energy is

$$E = \mu Mh + (16S_c\tilde{J}/M)\sin(Pa/4S_c) \tag{31}$$

and it satisfies the important relation

$$v = \partial E/\partial P = (4\tilde{J}a/M)\sin(Pa/4S_c). \tag{32}$$

Now let us look at the case of the weak anisotropy ($\tau \ll 1$). Equation (22) can be linearised in τ:

$$\begin{aligned} A_\xi^2 &= [(\gamma_0 + 2\tau/a^2 - V^2/4)A^2 - (\tfrac{1}{2}\gamma_0 + 2\tau/a^2)A^4 + (\tau/2a^2)A^6] \\ &\quad\times [1 + 2\tau A^2(1-\tfrac{1}{2}A^2)]^{-1} \\ &\simeq (\gamma_0 + 2\tau/a^2 - V^2/4)A^2 - [(\tfrac{1}{2}\gamma_0 + 2\tau/a^2)A^4 - (\tau/2a^2)A^6] \\ &\quad\times [1 + \mathrm{O}(\tau)] + \mathrm{O}(\tau^2 A^8) \\ &\simeq [\gamma_0 + 2\tau/a^2 - (V^2/4)A^2] - (\tfrac{1}{2}\gamma_0 + 2\tau/a^2)A^4 + (\tau/2a^2)A^6. \end{aligned} \tag{33}$$

This result agrees with our previous result (Škrinjar *et al* 1987) (except that in our previous paper τ appears instead τ/a^2; this is corrected here). Once again, we have arrived at the correct result with 'truncated' representation owing to exact cancelling of higher-order terms.

Still, it was important to perform here the complete study, because it was the only way to separate all classical solutions from the quantum corrections which will be studied in a subsequent paper.

Finally, we wish to note that equation (23) indicates in fact that Glauber's coherent states applied to the HP representation of spin operators along classical lines is completely equivalent to the application of the generalised (spin) coherent states (Perelomov 1985). We shall discuss this in more detail in the Appendix, but it is important to note here that this conclusion differs substantially from the conclusion of Makhankov *et al* (1987) and Makhankov and Makhankov (1988) who claim that the two approaches led to different equations in the continuum limit. Their conclusion is due to the truncation of the HP series and an incorrect continuum transition, because there is no way for the term of order $|\alpha|^6$ to appear in their calculation.

Appendix. Connection between Glauber's representation and generalised (spin) coherent states

In our approach (for $S \to \infty$), we have

$$\langle\alpha|S_j^+|\alpha\rangle = \sqrt{2S}\sqrt{1-\tfrac{1}{2}|\tilde{\alpha}_j|^2}\,\tilde{\alpha}_j$$

and since

$$\tilde{\alpha}_j \to A\exp(\mathrm{i}\varphi + \mathrm{i}\Omega t) \equiv A\exp(\mathrm{i}\phi_j)$$

and

$$\tfrac{1}{2}A^2 = \tfrac{1}{2}(1-\cos\theta)$$

we have

$$\langle\alpha|S_j^+|\alpha\rangle \to \sqrt{2S}\sqrt{1-\sin^2(\theta_j/2)}\sqrt{2}\sin(\theta_j/2)\exp(\mathrm{i}\phi_j) = S\sin\theta_j\exp(\mathrm{i}\phi_j).$$

In the same way,

$$\langle\alpha|S_j^-|\alpha\rangle \to S\sin\theta\exp(-\mathrm{i}\phi)$$

and

$$\langle\alpha|S_j^z|\alpha\rangle = S - |\alpha_j|^2 \to S(1-A^2) = S\cos\theta.$$

These average values are the same as those obtained by the application of the generalised coherent states (Perelomov 1985). This means that all equations which follow after averaging are the same, of course.

This implies that consequent application of the HP representation and Glauber's coherent states should lead precisely to the same results in the limit $S \to \infty$ as the application of generalised coherent states.

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